Fourier and Gegenbauer expansions for a fundamental solution of the Laplacian in the hyperboloid model of hyperbolic geometry

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Abstract. Due to the isotropy d-dimensional hyperbolic space, there exist a spherically symmetric fundamental solution for its corresponding Laplace-Beltrami operator. On the R-radius hyperboloid model of d-dimensional hyperbolic geometry with $R > 0$ and $d \geq 2$, we compute azimuthal Fourier expansions for a fundamental solution of Laplace's equation. For $d \geq 2$, we compute a Gegenbauer polynomial expansion in geodesic polar coordinates for a fundamental solution of Laplace's equation on this negative-constant sectional curvature Riemannian manifold. In three-dimensions, an addition theorem for the azimuthal Fourier coefficients of a fundamental solution for Laplace's equation is obtained through comparison with its corresponding Gegenbauer expansion.

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1. Introduction

In this paper we discuss eigenfunction expansions for a fundamental solution of Laplace's equation in the hyperboloid model of d-dimensional hyperbolic geometry. In particular, for a fixed $R \in (0, \infty)$ and $d \geq 2$, we derive and discuss Fourier cosine and Gegenbauer polynomial expansions in rotationally invariant coordinate systems, for a previously derived (see Cohl & Kalnins (2011) [8]) spherically symmetric fundamental solution of the Laplace-Beltrami operator in the hyperboloid model of hyperbolic geometry. Useful background material relevant for this paper can be found in Vilenkin (1968) [29], Thurston (1997) [27], Lee (1997) [20] and Pogosyan & Winternitz (2002) [26].

This paper is organized as follows. In section 2, for the hyperboloid model of d-dimensional hyperbolic geometry, we describe some of its global properties, such as its geodesic distance function, geodesic polar coordinates, Laplace-Beltrami operator (Laplacian), radial harmonics, and several previously derived equivalent expressions for a radial fundamental solution of Laplace's equation. In section 3, for $d \geq 2$, we derive and discuss Fourier cosine series for a fundamental solution of Laplace's equation about an appropriate azimuthal angle in rotationally invariant coordinate systems, and show how the resulting Fourier coefficients compare to the those in Euclidean space. In section 4, for $d \geq 2$, we compute Gegenbauer polynomial expansions in geodesic polar coordinates, for a fundamental solution of Laplace's equation in the hyperboloid model of hyperbolic geometry. In section 5 we discuss possible directions of research in this area.

Throughout this paper we rely on the following definitions. For $a_1, a_2, \ldots \in \mathbf{C}$, if $i, j \in \mathbf{Z}$ and $j < i$ then $\sum_{n=i}^{j} a_n = 0$ and $\prod_{n=i}^{j} a_n = 1$. The set of natural numbers is given by $\mathbf{N} := \{1, 2, \ldots\}$, the set $\mathbf{N}_0 := \{0, 1, 2, \ldots\} = \mathbf{N} \cup \{0\}$, and the set $\mathbf{Z} := \{0, \pm 1, \pm 2, \ldots\}$. The set $\mathbf{R}$ represents the real numbers.

2. Global analysis on the hyperboloid

2.1. The hyperboloid model of hyperbolic geometry

Hyperbolic space, developed independently by Lobachevsky and Bolyai around 1830 (see Trudeau (1987) [28]), is a fundamental example of a space exhibiting hyperbolic geometry. Hyperbolic geometry is analogous to Euclidean geometry, but such that Euclid's parallel postulate is no longer assumed to hold. There are several models of d-dimensional hyperbolic space $\mathbf{H}_R^d$, including the Klein, Poincaré, hyperboloid, upper-half space and hemisphere models (see Thurston (1997) [27]). The hyperboloid model for d-dimensional hyperbolic geometry (hereafter referred to as the hyperboloid model, or more simply, the hyperboloid), is closely related to the Klein and Poincaré models: each can be obtained projectively from the others. The upper-half space and hemisphere models can be obtained from one another by inversions with the Poincaré model (see section 2.2 in Thurston (1997) [27]). The model of hyperbolic geometry which we will be focusing on in this paper, is the hyperboloid model.

Minkowski space $\mathbf{R}^{d,1}$ is a $(d+1)$-dimensional pseudo-Riemannian manifold which is a real finite-dimensional vector space, with Cartesian coordinates given by $\mathbf{x} = (x_0, x_1, \ldots, x_d)$. The hyperboloid model, also known as the Minkowski or Lorentz models, represents points in this space by the upper sheet ($x_0 > 0$) of a two-sheeted hyperboloid embedded in the Minkowski space $\mathbf{R}^{d,1}$. It is equipped with a nondegenerate, symmetric bilinear form, the Minkowski bilinear form

$$[\mathbf{x}, \mathbf{y}] = x_0 y_0 - x_1 y_1 - \ldots - x_d y_d.$$

The above bilinear form is symmetric, but not positive-definite, so it is not an inner product. It is defined analogously with the Euclidean inner product for $\mathbf{R}^{d+1}$

$$(\mathbf{x}, \mathbf{y}) = x_0 y_0 + x_1 y_1 + \ldots + x_d y_d.$$

The variety $[\mathbf{x}, \mathbf{x}] = x_0^2 - x_1^2 - \ldots - x_d^2 = R^2$, for $\mathbf{x} \in \mathbf{R}^{d,1}$, using the language of Beltrami (1869) [3] (see also p. 504 in Vilenkin (1968) [29]), defines a pseudo-sphere of radius R. Points on the pseudo-sphere with zero radius coincide with a cone. Points on the pseudo-sphere with radius greater than zero lie within this cone, and points on the pseudo-sphere with purely imaginary radius lie outside the cone.

For a fixed $R \in (0, \infty)$, the R-radius hyperboloid model is a maximally symmetric, simply connected, d-dimensional Riemannian manifold with negative-constant sectional curvature (given by $-1/R^2$, see for instance p. 148 in Lee (1997) [20]), whereas Euclidean space $\mathbf{R}^d$ equipped with the Pythagorean norm, is a Riemannian manifold with zero sectional curvature. The hypersphere $\mathbf{S}^d$, is an example of a space (submanifold) with positive-constant sectional curvature (given by $1/R^2$).

In our discussion of a fundamental solution for Laplace's equation in the hyperboloid model $\mathbf{H}_R^d$, we focus on the positive radius pseudo-sphere which can be parametrized through *subgroup-type coordinates,* i.e. those which correspond to a maximal subgroup chain $O(d,1) \supset \ldots$ (see for instance Pogosyan & Winternitz (2002) [26]). There exist separable coordinate systems which parametrize points on the positive radius pseudo-sphere (i.e. such as those which are analogous to parabolic coordinates, etc.) which can not be constructed using maximal subgroup chains (we will no longer discuss these).

Geodesic polar coordinates are coordinates which correspond to the maximal subgroup chain given by $O(d,1) \supset O(d) \supset \ldots$. What we will refer to as *standard geodesic polar coordinates* correspond to the subgroup chain given by $O(d,1) \supset O(d) \supset O(d-1) \supset \cdots \supset O(2)$. Standard geodesic polar coordinates (see Olevskiĭ (1950) [23]; Grosche, Pogosyan & Sissakian (1997) [16]), similar to standard hyperspherical

coordinates in Euclidean space, can be given by

$$\left.\begin{array}{rcl} x_0 &=& R\cosh r \\ x_1 &=& R\sinh r\cos\theta_1 \\ x_2 &=& R\sinh r\sin\theta_1\cos\theta_2 \\ &\vdots& \\ x_{d-2} &=& R\sinh r\sin\theta_1\cdots\cos\theta_{d-2} \\ x_{d-1} &=& R\sinh r\sin\theta_1\cdots\sin\theta_{d-2}\cos\phi \\ x_d &=& R\sinh r\sin\theta_1\cdots\sin\theta_{d-2}\sin\phi, \end{array}\right\} \qquad (1)$$

where $r\in[0,\infty)$, $\phi\in[0,2\pi)$, and $\theta_i\in[0,\pi]$ for $i\in\{1,\ldots,d-2\}$.

In order to study fundamental solutions on the hyperboloid, we need to describe how one computes distances in this space. One may naturally compare distances on the positive radius pseudo-sphere through analogy with the R-radius hypersphere. Distances on the hypersphere are simply given by arc lengths, angles between two arbitrary vectors, from the origin, in the ambient Euclidean space. We consider the d-dimensional hypersphere embedded in $\mathbf{R}^{d+1}$. Points on the hypersphere can be parametrized using hyperspherical coordinate systems. Any parametrization of the hypersphere $\mathbf{S}^d$, must have $(\mathbf{x},\mathbf{x})=x_0^2+\ldots+x_d^2=R^2$, with $R>0$. The distance between two points on the hypersphere is given by

$$d(\mathbf{x},\mathbf{x}')=R\gamma=R\cos^{-1}\left(\frac{(\mathbf{x},\mathbf{x}')}{(\mathbf{x},\mathbf{x})(\mathbf{x}',\mathbf{x}')}\right)=R\cos^{-1}\left(\frac{1}{R^2}(\mathbf{x},\mathbf{x}')\right). \qquad (2)$$

This is evident from the fact that the geodesics on $\mathbf{S}^d$ are great circles (i.e. intersections of $\mathbf{S}^d$ with planes through the origin) with constant speed parametrizations (see p. 82 in Lee (1997) [20]).

Accordingly, we now look at the geodesic distance function on the d-dimensional positive radius pseudo-sphere $\mathbf{H}_R^d$. Distances between two points on the positive radius pseudo-sphere are given by the hyperangle between two arbitrary vectors, from the origin, in the ambient Minkowski space. Any parametrization of the hyperboloid $\mathbf{H}_R^d$, must have $[\mathbf{x},\mathbf{x}]=R^2$. The geodesic distance $\rho\in[0,\infty)$ between two points $\mathbf{x},\mathbf{x}'\in\mathbf{H}_R^d$ is given by

$$d(\mathbf{x},\mathbf{x}')=R\cosh^{-1}\left(\frac{[\mathbf{x},\mathbf{x}']}{[\mathbf{x},\mathbf{x}][\mathbf{x}',\mathbf{x}']}\right)=R\cosh^{-1}\left(\frac{1}{R^2}[\mathbf{x},\mathbf{x}']\right), \qquad (3)$$

where the inverse hyperbolic cosine with argument $x\in(1,\infty)$ is given by (see (4.37.19) in Olver *et al.* (2010) [25]) $\cosh^{-1}x=\log\left(x+\sqrt{x^2-1}\right)$. Geodesics on $\mathbf{H}_R^d$ are great hyperbolas (i.e. intersections of $\mathbf{H}_R^d$ with planes through the origin) with constant speed parametrizations (see p. 84 in Lee (1997) [20]). We also define a global function $\rho:\mathbf{H}^d\times\mathbf{H}^d\to[0,\infty)$ which represents the projection of global geodesic distance function (3) on $\mathbf{H}_R^d$ onto the corresponding unit radius hyperboloid $\mathbf{H}^d$, namely

$$\rho(\widehat{\mathbf{x}},\widehat{\mathbf{x}}'):=d(\mathbf{x},\mathbf{x}')/R, \qquad (4)$$

where $\widehat{\mathbf{x}}=\mathbf{x}/R$ and $\widehat{\mathbf{x}}'=\mathbf{x}'/R$. Note that when we refer to $d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}')$ below, we specifically mean that projected distance given by (4).

2.2. Laplace's equation and harmonics on the hyperboloid

Parametrizations of a submanifold embedded in either a Euclidean or Minkowski space is given in terms of coordinate systems whose coordinates are curvilinear. These are coordinates based on some transformation that converts the standard Cartesian coordinates in the ambient space to a coordinate system with the same number of coordinates as the dimension of the submanifold in which the coordinate lines are curved.

The Laplace-Beltrami operator (Laplacian) in curvilinear coordinates $\xi = (\xi^1, \ldots, \xi^d)$ on a d-dimensional Riemannian manifold (a manifold together with a Riemannian metric g) is given by

$$\Delta = \sum_{i,j=1}^{d} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \xi^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial \xi^j} \right), \tag{5}$$

where $|g| = |\det(g_{ij})|$, the infinitesimal distance is given by

$$ds^2 = \sum_{i,j=1}^{d} g_{ij} d\xi^i d\xi^j, \tag{6}$$

and

$$\sum_{i=1}^{d} g_{ki} g^{ij} = \delta_k^j,$$

where $\delta_i^j \in \{0, 1\}$ with $i, j \in \mathbf{Z}$, is the Kronecker delta defined such that

$$\delta_i^j := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{7}$$

For a submanifold, the relation between the metric tensor in the ambient space and g_{ij} of (5) and (6) is

$$g_{ij}(\xi) = \sum_{k,l=0}^{d} G_{kl} \frac{\partial x^k}{\partial \xi^i} \frac{\partial x^l}{\partial \xi^j}.$$

The ambient space for the hyperboloid is Minkowski, and therefore $G_{ij} = \mathrm{diag}(1, -1, \ldots, -1)$.

The set of all geodesic polar coordinate systems corresponds to the many ways one can put coordinates on a hyperbolic hypersphere, i.e., the Riemannian submanifold $U \subset \mathbf{H}_R^d$ defined for a fixed $\mathbf{x}' \in \mathbf{H}_R^d$ such that $d(\mathbf{x}, \mathbf{x}') = b = const$, where $b \in (0, \infty)$. These are coordinate systems which correspond to subgroup chains starting with $O(d, 1) \supset O(d) \supset \cdots$, with standard geodesic polar coordinates given by (1) being only one of them. (For a thorough description of these see section X.5 in Vilenkin (1968) [29].) They all share the property that they are described by $(d+1)$-variables: $r \in [0, \infty)$ plus d-angles each being given by the values $[0, 2\pi)$, $[0, \pi]$, $[-\pi/2, \pi/2]$ or $[0, \pi/2]$ (see Izmest'ev *et al.* (1999, 2001) [17, 18]).

In any of the geodesic polar coordinate systems, the geodesic distance between two points on the submanifold is given by (cf. (3))

$$d(\mathbf{x}, \mathbf{x}') = R \cosh^{-1}(\cosh r \cosh r' - \sinh r \sinh r' \cos \gamma), \tag{8}$$

where γ is the unique separation angle given in each hyperspherical coordinate system. For instance, the separation angle in standard geodesic polar coordinates (1) is given by the formula

$$\cos\gamma = \cos(\phi-\phi')\prod_{i=1}^{d-2}\sin\theta_i\sin\theta_i' + \sum_{i=1}^{d-2}\cos\theta_i\cos\theta_i'\prod_{j=1}^{i-1}\sin\theta_j\sin\theta_j'. \tag{9}$$

Corresponding separation angle formulae for any geodesic polar coordinate system can be computed using (2), (3), and the associated formulae for the appropriate inner-products.

The infinitesimal distance in a geodesic polar coordinate system on this submanifold is

$$ds^2 = R^2(dr^2 + \sinh^2 r\ d\gamma^2), \tag{10}$$

where an appropriate expression for γ in a curvilinear coordinate system is given. If one combines (1), (5), (9) and (10), then in a particular geodesic polar coordinate system, Laplace's equation on $\mathbf{H}_R^d$ is

$$\Delta f = \frac{1}{R^2}\left[\frac{\partial^2 f}{\partial r^2} + (d-1)\coth r\frac{\partial f}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{\mathbf{S}^{d-1}} f\right] = 0, \tag{11}$$

where $\Delta_{\mathbf{S}^{d-1}}$ is the corresponding Laplace-Beltrami operator on $\mathbf{S}^{d-1}$ with unit radius.

From this point onwards, $\mathbf{S}^{d-1}$ will always refer to the $(d-1)$-dimensional unit hypersphere, which is a compact Riemannian submanifold with positive constant sectional curvature, embedded in $\mathbf{R}^d$ and given by the variety $x_1^2+\ldots+x_d^2=1$.

Geodesic polar coordinate systems partition $\mathbf{H}_R^d$ into a family of concentric $(d-1)$-dimensional hyperspheres, each with a radius $r\in(0,\infty)$, on which all possible hyperspherical coordinate systems for $\mathbf{S}^{d-1}$ may be used (see for instance, in Vilenkin (1968) [29]). One then must also consider the limiting case for $r=0$ to fill out all of $\mathbf{H}_R^d$. In standard geodesic polar coordinates one can compute the normalized hyperspherical harmonics in this space by solving the Laplace equation using separation of variables which results in a general procedure which is given explicitly in Izmest'ev *et al.* (1999, 2001) [17, 18]. These angular harmonics are given as general expressions involving trigonometric functions, Gegenbauer polynomials and Jacobi polynomials.

The harmonics in geodesic polar coordinate systems are given in terms of a radial solution multiplied by the angular harmonics. The angular harmonics are eigenfunctions of the Laplace-Beltrami operator on $\mathbf{S}^{d-1}$ with unit radius which satisfy the following eigenvalue problem

$$\Delta_{\mathbf{S}^{d-1}} Y_l^K(\widehat{\mathbf{x}}) = -l(l+d-2)Y_l^K(\widehat{\mathbf{x}}), \tag{12}$$

where $\widehat{\mathbf{x}}\in\mathbf{S}^{d-1}$, $Y_l^K(\widehat{\mathbf{x}})$ are normalized hyperspherical harmonics, $l\in\mathbf{N}_0$ is the angular momentum quantum number, and K stands for the set of $(d-2)$-quantum numbers identifying degenerate harmonics for each l. The degeneracy as a function of the

dimension d tells you how many linearly independent solutions exist for a particular l value. The hyperspherical harmonics are normalized such that

$$\int_{\mathbf{S}^{d-1}} Y_l^K(\widehat{\mathbf{x}})\overline{Y_{l'}^{K'}(\widehat{\mathbf{x}})}d\omega = \delta_l^{l'}\delta_K^{K'},$$

where $d\omega$ is a volume measure on $\mathbf{S}^{d-1}$ which is invariant under the isometry group $SO(d)$ (cf. (14)), and for $x+iy=z\in\mathbf{C}$, $\overline{z}=x-iy$, represents complex conjugation. The generalized Kronecker delta $\delta_K^{K'}$ (cf. (7)) is defined such that it equals 1 if all of the $(d-2)$-quantum numbers identifying degenerate harmonics for each l coincide, and equals zero otherwise.

Since the angular solutions (hyperspherical harmonics) are well-known (see Chapter IX in Vilenkin (1968) [29]; Chapter 11 in Erdélyi *et al.* (1981) [12]), we will now focus on the radial solutions, which satisfy the following ordinary differential equation

$$\frac{d^2u}{dr^2} + (d-1)\coth r\frac{du}{dr} - \frac{l(l+d-2)}{\sinh^2 r}u = 0.$$

Four solutions to this ordinary differential equation $u_{1\pm}^{d,l}, u_{2\pm}^{d,l} : (1\infty) \to \mathbf{C}$ are given by

$$u_{1\pm}^{d,l}(\cosh r) = \frac{1}{\sinh^{d/2-1} r}P_{d/2-1}^{\pm(d/2-1+l)}(\cosh r),$$

and

$$u_{2\pm}^{d,l}(\cosh r) = \frac{1}{\sinh^{d/2-1} r}Q_{d/2-1}^{\pm(d/2-1+l)}(\cosh r),$$

where $P_\nu^\mu, Q_\nu^\mu : (1,\infty) \to \mathbf{C}$ are associated Legendre functions of the first and second kind respectively (see for instance Chapter 14 in Olver *et al.* (2010) [25]).

2.3. Fundamental solution of Laplace's equation on the hyperboloid

Due to the fact that the space $\mathbf{H}_R^d$ is homogeneous with respect to its isometry group, the pseudo-orthogonal group $SO(d,1)$, and therefore an isotropic manifold, we expect that there exist a fundamental solution on this space with spherically symmetric dependence. We specifically expect these solutions to be given in terms of associated Legendre functions of the second kind with argument given by $\cosh r$. This associated Legendre function naturally fits our requirements because it is singular at $r=0$ and vanishes at infinity, whereas the associated Legendre functions of the first kind, with the same argument, are regular at $r=0$ and singular at infinity.

In computing a fundamental solution of the Laplacian on $\mathbf{H}_R^d$, we know that

$$-\Delta\mathcal{H}_R^d(\mathbf{x},\mathbf{x}') = \delta_g(\mathbf{x},\mathbf{x}'), \tag{13}$$

where g is the Riemannian metric on $\mathbf{H}_R^d$ and $\delta_g(\mathbf{x},\mathbf{x}')$ is the Dirac delta function on the manifold $\mathbf{H}_R^d$. The Dirac delta function is defined for an open set $U\subset\mathbf{H}_R^d$ with $\mathbf{x},\mathbf{x}'\in\mathbf{H}_R^d$ such that

$$\int_U \delta_g(\mathbf{x},\mathbf{x}')d\mathrm{vol}_g = \begin{cases} 1 & \text{if } \mathbf{x}'\in U, \\ 0 & \text{if } \mathbf{x}'\notin U, \end{cases}$$

where $d\mathrm{vol}_g$ is a volume measure, invariant under the isometry group $SO(d,1)$ of the Riemannian manifold $\mathbf{H}_R^d$, given (in standard geodesic polar coordinates) by

$$d\mathrm{vol}_g = R^d \sinh^{d-1} r d\omega := R^d \sinh^{d-1} r \sin^{d-2}\theta_{d-1}\cdots\sin\theta_2 d\theta_1\cdots d\theta_{d-1}. \tag{14}$$

Notice that as $r \to 0^+$ that $d\mathrm{vol}_g$ goes to the Euclidean measure, invariant under the Euclidean motion group $E(d)$, in spherical coordinates. Therefore in spherical coordinates, we have the following

$$\delta_g(\mathbf{x},\mathbf{x}') = \frac{\delta(r-r')}{R^d\sinh^{d-1} r'}\frac{\delta(\theta_1-\theta_1')\cdots\delta(\theta_{d-1}-\theta_{d-1}')}{\sin\theta_2'\cdots\sin^{d-2}\theta_{d-1}'}. \tag{15}$$

In general since we can add any harmonic function to a fundamental solution for the Laplacian and still have a fundamental solution, we will use this freedom to make our fundamental solution as simple as possible. It is reasonable to expect that there exists a particular spherically symmetric fundamental solution $\mathcal{H}_R^d(\mathbf{x},\mathbf{x}')$ on the hyperboloid with pure radial $\rho(\widehat{\mathbf{x}},\widehat{\mathbf{x}}') = d(\mathbf{x},\mathbf{x}')/R$ (cf. (4)) and constant angular dependence (invariant under rotations centered about the origin), due to the influence of the point-like nature of the Dirac delta function. For a spherically symmetric solution to the Laplace equation, the corresponding $\Delta_{\mathbf{S}^{d-1}}$ term vanishes since only the $l=0$ term survives. In other words, we expect there to exist a fundamental solution of Laplace's equation such that $\mathcal{H}_R^d(\mathbf{x},\mathbf{x}') = f(\rho)$.

In Cohl & Kalnins (2011) [8], we have proven that on the R-radius hyperboloid $\mathbf{H}_R^d$, a fundamental solution of Laplace's equation can be given as follows.

Theorem 2.1. *Let $d\in\{2,3,\ldots\}$. Define $\mathcal{I}_d:(0,\infty)\to\mathbf{R}$ as*

$$\mathcal{I}_d(\rho) := \int_\rho^\infty \frac{dx}{\sinh^{d-1}x},$$

$\mathbf{x},\mathbf{x}'\in\mathbf{H}_R^d$, and $\mathcal{H}_R^d : (\mathbf{H}_R^d\times\mathbf{H}_R^d)\setminus\{(\mathbf{x},\mathbf{x}):\mathbf{x}\in\mathbf{H}_R^d\}\to\mathbf{R}$ defined such that

$$\mathcal{H}_R^d(\mathbf{x},\mathbf{x}') := \frac{\Gamma(d/2)}{2\pi^{d/2}R^{d-2}}\mathcal{I}_d(\rho),$$

where $\rho := \cosh^{-1}([\widehat{\mathbf{x}},\widehat{\mathbf{x}}'])$ is the geodesic distance between $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{x}}'$ on the pseudo-sphere of unit radius $\mathbf{H}^d$, with $\widehat{\mathbf{x}} = \mathbf{x}/R$, $\widehat{\mathbf{x}}' = \mathbf{x}'/R$, then $\mathcal{H}_R^d$ is a fundamental solution for $-\Delta$ where Δ is the Laplace-Beltrami operator on $\mathbf{H}_R^d$. Moreover,

$$\mathcal{I}_d(\rho) = \begin{cases} (-1)^{d/2-1}\dfrac{(d-3)!!}{(d-2)!!}\left[\log\coth\dfrac{\rho}{2} + \cosh\rho\displaystyle\sum_{k=1}^{d/2-1}\frac{(2k-2)!!(-1)^k}{(2k-1)!!\sinh^{2k}\rho}\right] & \text{if } d \text{ even,} \\ \left\{\begin{array}{l} (-1)^{(d-1)/2}\left[\dfrac{(d-3)!!}{(d-2)!!}\right. \\ \qquad\left. + \left(\dfrac{d-3}{2}\right)! \displaystyle\sum_{k=1}^{(d-1)/2}\frac{(-1)^k\coth^{2k-1}\rho}{(2k-1)(k-1)!((d-2k-1)/2)!}\right], \\ \text{or} \\ (-1)^{(d-1)/2}\dfrac{(d-3)!!}{(d-2)!!}\left[1+\cosh\rho\displaystyle\sum_{k=1}^{(d-1)/2}\frac{(2k-3)!!(-1)^k}{(2k-2)!!\sinh^{2k-1}\rho}\right], \end{array}\right\} & \text{if } d \text{ odd.} \end{cases}$$

$$= \frac{1}{(d-1)\cosh^{d-1}\rho}\,{}_2F_1\left(\frac{d-1}{2},\frac{d}{2};\frac{d+1}{2};\frac{1}{\cosh^2\rho}\right),$$

$$= \frac{1}{(d-1)\cosh\rho\,\sinh^{d-2}\rho}\,{}_2F_1\left(\frac{1}{2},1;\frac{d+1}{2};\frac{1}{\cosh^2\rho}\right),$$

$$= \frac{e^{-i\pi(d/2-1)}}{2^{d/2-1}\Gamma\left(d/2\right)\sinh^{d/2-1}\rho}\,Q_{d/2-1}^{d/2-1}(\cosh\rho),$$

where !! *is the double factorial,* ${}_2F_1$ *is the Gauss hypergeometric function, and* Q_ν^μ *is the associated Legendre function of the second kind.*

For a proof of this theorem, see Cohl & Kalnins (2011) [8].

3. Fourier expansions for a Green's function on the hyperboloid

Now we compute the Fourier expansions for a fundamental solution of the Laplace-Beltrami operator on $\mathbf{H}_R^d$.

3.1. Fourier expansion for a fundamental solution of the Laplacian on $\mathbf{H}_R^2$

The generating function for Chebyshev polynomials of the first kind (Fox & Parker (1968) [14], p. 51) is given as

$$\frac{1-z^2}{1+z^2-2xz} = \sum_{n=0}^{\infty}\epsilon_n T_n(x) z^n, \tag{16}$$

where $|z|<1$, $T_n:[-1,1]\to\mathbf{R}$ is the Chebyshev polynomial of the first kind defined as $T_l(x) := \cos(l\cos^{-1}x)$, and $\epsilon_n := 2-\delta_n^0$ is the Neumann factor (see p. 744 in Morse & Feshbach (1953) [22]), commonly-occurring in Fourier cosine series. If substitute $z=e^{-\eta}$ with $\eta\in(0,\infty)$ in (16), then we obtain

$$\frac{\sinh\eta}{\cosh\eta-\cos\psi} = \sum_{n=0}^{\infty}\epsilon_n\cos(n\psi)e^{-n\eta}. \tag{17}$$

Integrating both sides of (17) with respect to η, we obtain the following formula (cf. Magnus, Oberhettinger & Soni (1966) [21], p. 259)

$$\log\left(1+z^2-2z\cos\psi\right) = -2\sum_{n=1}^{\infty}\frac{\cos(n\psi)}{n}z^n. \tag{18}$$

In Euclidean space $\mathbf{R}^d$, a Green's function for Laplace's equation (fundamental solution for the Laplacian) is well-known and is given in the following theorem (see Folland (1976) [13]; p. 94, Gilbarg & Trudinger (1983) [15]; p. 17, Bers *et al.* (1964) [4], p. 211).

Theorem 3.1. *Let $d \in \mathbf{N}$. Define*

$$\mathcal{G}^d(\mathbf{x},\mathbf{x}') = \begin{cases} \dfrac{\Gamma(d/2)}{2\pi^{d/2}(d-2)}\|\mathbf{x}-\mathbf{x}'\|^{2-d} & \text{if } d=1 \text{ or } d\ge 3,\\ \dfrac{1}{2\pi}\log\|\mathbf{x}-\mathbf{x}'\|^{-1} & \text{if } d=2,\end{cases}$$

then $\mathcal{G}^d$ is a fundamental solution for $-\Delta$ in Euclidean space $\mathbf{R}^d$, where Δ is the Laplace operator in $\mathbf{R}^d$.

Therefore if we take $z = r_</r_>$ in (18), where $r_{\lessgtr} := {\min \atop \max}\{r, r'\}$ with $r, r' \in [0,\infty)$, then using polar coordinates, we can derive the Fourier expansion for a fundamental solution of the Laplacian in Euclidean space for $d = 2$ (cf. Theorem 3.1), namely

$$\mathfrak{g}^2 := \log\|\mathbf{x}-\mathbf{x}'\| = \log r_> - \sum_{n=1}^{\infty}\frac{\cos(n(\phi-\phi'))}{n}\left(\frac{r_<}{r_>}\right)^n, \tag{19}$$

where $\mathfrak{g}^2 = -2\pi\mathcal{G}^2$ (cf. Theorem 3.1). On the hyperboloid for $d = 2$ we have a fundamental solution of Laplace's equation given by

$$\mathfrak{h}^2 := \log\coth\frac{1}{2}d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}') = \frac{1}{2}\log\frac{\cosh d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}')+1}{\cosh d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}')-1},$$

where $\mathfrak{h}^2 = 2\pi\mathcal{H}_R^2$ (cf. Theorem 2.1 and (31) below). Note that because of the R^{d-2} dependence of a fundamental solution of Laplace's equation for $d = 2$ in Theorem 2.1, there is no strict dependence on R for $\mathcal{H}_R^2$ or $\mathfrak{h}^2$, but will retain the notation nonetheless. In standard geodesic polar coordinates on $\mathbf{H}_R^2$ (cf. (1)), using (8) and $\cos\gamma = \cos(\phi-\phi')$ (cf. (9)) produces

$$\cosh d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}') = \cosh r\cosh r' - \sinh r\sinh r'\cos(\phi-\phi'),$$

therefore

$$\mathfrak{h}^2 = \frac{1}{2}\log\frac{\cosh r\cosh r' + 1 - \sinh r\sinh r'\cos(\phi-\phi')}{\cosh r\cosh r' - 1 - \sinh r\sinh r'\cos(\phi-\phi')}.$$

Replacing $\psi = \phi - \phi'$ and rearranging the logarithms yield

$$\mathfrak{h}^2 = \frac{1}{2}\log\frac{\cosh r\cosh r'+1}{\cosh r\cosh r'-1} + \frac{1}{2}\log\left(1 - z_+\cos\psi\right) - \frac{1}{2}\log\left(1 - z_-\cos\psi\right),$$

where

$$z_\pm := \frac{\sinh r\sinh r'}{\cosh r\cosh r'\pm 1}.$$

Note that $z_\pm \in (0,1)$ for $r, r' \in (0,\infty)$. We have the following MacLaurin series

$$\log(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n},$$

where $x \in [-1, 1)$. Therefore away from the singularity at $\mathbf{x} = \mathbf{x}'$ we have

$$\lambda_\pm := \log\left(1 - z_\pm\cos\psi\right) = -\sum_{k=1}^{\infty}\frac{z_\pm^k}{k}\cos^k\psi. \tag{20}$$

We can expand the powers of cosine using the following trigonometric identity

$$\cos^k \psi = \frac{1}{2^k} \sum_{n=0}^{k} \binom{k}{n} \cos[(2n-k)\psi],$$

which is the standard expansion for powers using Chebyshev polynomials (see for instance p. 52 in Fox & Parker (1968) [14]). Inserting this expression in (20), we obtain the following double-summation expression

$$\lambda_\pm = -\sum_{k=1}^{\infty} \sum_{n=0}^{k} \frac{z_\pm^k}{2^k k} \binom{k}{n} \cos[(2n-k)\psi]. \tag{21}$$

Now we perform a double-index replacement in (21). We break this sum into two separate sums, one for $k \le 2n$ and another for $k \ge 2n$. There is an overlap when both sums satisfy the equality, and in that situation we must halve after we sum over both sums. If $k \le 2n$, make the substitution $k' = k - n$ and $n' = 2n - k$. It follows that $k = 2k' + n'$ and $n = n' + k'$, therefore

$$\binom{k}{n} = \binom{2k'+n'}{n'+k'} = \binom{2k'+n'}{n'+k'}.$$

If $k \ge 2n$ make the substitution $k' = n$ and $n' = k - 2n$. Then $k = 2k' + n'$ and $n = k'$, therefore

$$\binom{k}{n} = \binom{2k'+n'}{n} = \binom{2k'+n'}{k'+n'},$$

where the equalities of the binomial coefficients are confirmed using the following identity

$$\binom{n}{k} = \binom{n}{n-k},$$

where $n, k \in \mathbf{Z}$, except where $k < 0$ or $n - k < 0$. To take into account the double-counting which occurs when $k = 2n$ (which occurs when $n' = 0$), we introduce a factor of $\epsilon_{n'}/2$ into the expression (and relabel $k' \mapsto k$ and $n' \mapsto n$). We are left with

$$\lambda_\pm = -\frac{1}{2} \sum_{k=1}^{\infty} \frac{z_\pm^{2k}}{2^{2k} k} \binom{2k}{k} - 2 \sum_{n=1}^{\infty} \cos(n\psi) \sum_{k=0}^{\infty} \frac{z_\pm^{2k+n}}{2^{2k+n}(2k+n)} \binom{2k+n}{k}. \tag{22}$$

If we substitute

$$\binom{2k}{k} = \frac{2^{2k} \left(\frac{1}{2}\right)_k}{k!}$$

into the first term of (22), then we obtain

$$I_\pm := -\frac{1}{2} \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)_k z_\pm^{2k}}{k! k} = -\int_0^{z_\pm} \frac{dz'_\pm}{z'_\pm} \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)_k {z'_\pm}^{2k}}{k!} = -\int_0^{z_\pm} \frac{dz'_\pm}{z'_\pm} \left[\frac{1}{\sqrt{1 - {z'_\pm}^2}} - 1 \right].$$

We are left with

$$I_\pm = -\log 2 + \log\left(1 + \sqrt{1 - z_\pm^2}\right) = -\log 2 + \log\left(\frac{(\cosh r_> \pm 1)(\cosh r_< + 1)}{\cosh r \cosh r' \pm 1}\right).$$

If we substitute

$$\binom{2k+n}{k} = \frac{2^{2k}\left(\frac{n+1}{2}\right)_k\left(\frac{n+2}{2}\right)_k}{k!(n+1)_k},$$

into the second term of (22), then the Fourier coefficient reduces to

$$J_\pm := \frac{1}{2^{n-1}}\sum_{k=0}^{\infty}\frac{\left(\frac{n+1}{2}\right)_k\left(\frac{n+2}{2}\right)_k}{k!(n+1)_k}\frac{z_\pm^{2k+n}}{2k+n}$$

$$= \frac{1}{2^{n-1}}\int_0^{z_\pm} dz'_\pm {z'_\pm}^{n-1}\sum_{k=0}^{\infty}\frac{\left(\frac{n+1}{2}\right)_k\left(\frac{n+2}{2}\right)_k}{k!(n+1)_k}{z'_\pm}^{2k}.$$

The series in the integrand is a Gauss hypergeometric function which can be given as

$$\sum_{k=0}^{\infty}\frac{\left(\frac{n+1}{2}\right)_k\left(\frac{n+2}{2}\right)_k}{k!(n+1)_k}z^{2k} = \frac{2^n n!}{z^n\sqrt{1-z^2}}P_0^{-n}\left(\sqrt{1-z^2}\right),$$

where P_0^{-n} is an associated Legendre function of the first kind with vanishing degree and order given by $-n$. This is a consequence of

$${}_2F_1\left(a,b;a+b-\frac{1}{2};x\right) = 2^{2+b-3/2}\Gamma\left(a+b-\frac{1}{2}\right)\frac{x^{(3-2a-2b)/4}}{\sqrt{1-x}}P_{b-a-1/2}^{3/2-a-b}\left(\sqrt{1-x}\right),$$

where $x \in (0,1)$ (see for instance Magnus, Oberhettinger & Soni (1966) [21], p. 53), and the Legendre function is evaluated using (cf. (8.1.2) in Abramowitz & Stegun (1972) [1])

$$P_0^{-n}(x) = \frac{1}{n!}\left(\frac{1-x}{1+x}\right)^{n/2},$$

where $n \in \mathbf{N}_0$. Therefore the Fourier coefficient is given by

$$J_\pm = 2\int_{\sqrt{1-z_\pm^2}}^{1}\frac{dz'_\pm}{1-{z'_\pm}^2}\left(\frac{1-z'_\pm}{1+z'_\pm}\right)^{n/2} = \frac{2}{n}\left[\frac{1-\sqrt{1-z_\pm^2}}{1+\sqrt{1-z_\pm^2}}\right]^{n/2}.$$

Finally we have

$$\lambda_\pm = -\log 2 + \log\left(\frac{(\cosh r_> \pm 1)(\cosh r_< + 1)}{\cosh r\cosh r' \pm 1}\right)$$
$$-2\sum_{n=1}^{\infty}\frac{\cos(n\psi)}{n}\left[\frac{(\cosh r_> \mp 1)(\cosh r_< - 1)}{(\cosh r_> \pm 1)(\cosh r_< + 1)}\right]^{n/2},$$

and the Fourier expansion for a fundamental solution of Laplace's equation for the $d=2$ hyperboloid is given by

$$\mathfrak{h}^2 = \frac{1}{2}\log\frac{\cosh r_> + 1}{\cosh r_> - 1}$$
$$+\sum_{n=1}^{\infty}\frac{\cos(n(\phi-\phi'))}{n}\left[\frac{\cosh r_< - 1}{\cosh r_< + 1}\right]^{n/2}\left\{\left[\frac{\cosh r_> + 1}{\cosh r_> - 1}\right]^{n/2} - \left[\frac{\cosh r_> - 1}{\cosh r_> + 1}\right]^{n/2}\right\}. \quad (23)$$

This exactly matches up to the Euclidean Fourier expansion $\mathfrak{g}^2$ (19) as $r, r' \to 0^+$.

3.2. Fourier expansion for a fundamental solution of the Laplacian on $\mathbf{H}_R^3$

The Fourier expansion for a fundamental solution of the Laplacian in three-dimensional Euclidean space (here given in standard spherical coordinates $\mathbf{x} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$) is given by (cf. Theorem 3.1, and see (1.3) in Cohl *et al.* (2001) [9])

$$\mathcal{G}^3 \simeq \mathfrak{g}^3 := \frac{1}{\|\mathbf{x}-\mathbf{x}'\|} = \frac{1}{\pi\sqrt{rr'\sin\theta\sin\theta'}}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}Q_{m-1/2}\left(\frac{r^2+r'^2-2rr'\cos\theta\cos\theta'}{2rr'\sin\theta\sin\theta'}\right).$$

These associated Legendre functions, toroidal harmonics, are given in terms of complete elliptic integrals of the first and second kind (cf. (22–26) in Cohl & Tohline (1999) [10]). Since $Q_{-1/2}(z)$ is given through (cf. (8.13.3) in Abramowitz & Stegun (1972) [1])

$$Q_{-1/2}(z) = \sqrt{\frac{2}{z+1}}K\left(\sqrt{\frac{2}{z+1}}\right),$$

the $m = 0$ component for $\mathfrak{g}^3$ is given by

$$\mathfrak{g}^3\Big|_{m=0} = \frac{2}{\pi\sqrt{r^2+r'^2-2rr'\cos(\theta+\theta')}}K\left(\sqrt{\frac{4rr'\sin\theta\sin\theta'}{r^2+r'^2-2rr'\cos(\theta+\theta')}}\right). \tag{24}$$

A fundamental solution of the Laplacian in standard geodesic polar coordinates on $\mathbf{H}_R^3$ is given by (cf. Theorem 2.1 and (31) below).

$$\mathfrak{h}^3(\widehat{\mathbf{x}},\widehat{\mathbf{x}}') := \coth d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}') - 1 = \frac{\cosh d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}')}{\sqrt{\cosh^2 d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}')-1}} - 1 = \frac{\cosh r\cosh r' - \sinh r\sinh r'\cos\gamma}{\sqrt{(\cosh r\cosh r' - \sinh r\sinh r'\cos\gamma)^2-1}} - 1,$$

where $\mathfrak{h}^3 = 4\pi R\mathcal{H}_R^3$, and $\mathbf{x},\mathbf{x}' \in \mathbf{H}_R^3$, such that $\widehat{\mathbf{x}} = \mathbf{x}/R$ and $\widehat{\mathbf{x}}' = \mathbf{x}'/R$. In standard geodesic polar coordinates (cf. (9)) we have

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'). \tag{25}$$

Replacing $\psi = \phi - \phi'$ and defining

$$A := \cosh r\cosh r' - \sinh r\sinh r'\cos\theta\cos\theta',$$

and

$$B := \sinh r\sinh r'\sin\theta\sin\theta',$$

we have in the standard manner, the Fourier coefficients $\mathsf{H}_m^{1/2} : [0,\infty)^2\times[0,\pi]^2 \to \mathbf{R}$ of the expansion (cf. (32) below)

$$\mathfrak{h}^3(\widehat{\mathbf{x}},\widehat{\mathbf{x}}') = \sum_{m=0}^{\infty}\cos(m(\phi-\phi'))\mathsf{H}_m^{1/2}(r,r',\theta,\theta'), \tag{26}$$

defined by

$$\mathsf{H}_m^{1/2}(r,r',\theta,\theta') := -\delta_n^0 + \frac{\epsilon_m}{\pi}\int_0^\pi \frac{(A/B-\cos\psi)\cos(m\psi)d\psi}{\sqrt{\left(\cos\psi-\frac{A+1}{B}\right)\left(\cos\psi-\frac{A-1}{B}\right)}}. \tag{27}$$

If we make the substitution $x=\cos\psi$, this integral can be converted to

$$\mathsf{H}_m^{1/2}(r,r',\theta,\theta') = -\delta_n^0 + \frac{\epsilon_m}{\pi}\int_{-1}^1 \frac{(A/B-x)\,T_m(x)dx}{\sqrt{(1-x)(1+x)\left(x-\frac{A+1}{B}\right)\left(x-\frac{A-1}{B}\right)}}, \tag{28}$$

where T_m is the Chebyshev polynomial of the first kind. Since $T_m(x)$ is expressible as a finite sum over powers of x, (28) involves the square root of a quartic multiplied by a rational function of x, which by definition is an elliptic integral (see for instance Byrd & Friedman (1954) [5]). We can directly compute (28) using Byrd & Friedman (1954) ([5], (253.11)). If we define

$$d:=-1,\ y:=-1,\ c:=1,\ b:=\frac{A-1}{B},\ a:=\frac{A+1}{B}, \tag{29}$$

(clearly $d\le y<c<b<a$), then we can express the Fourier coefficient (28), as a linear combination of integrals, each of the form (see Byrd & Friedman (1954) [5], (253.11))

$$\int_y^c \frac{x^p dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = c^p g\int_0^{u_1}\left[\frac{1-\alpha_1^2\mathrm{sn}^2u}{1-\alpha^2\mathrm{sn}^2u}\right]^p du, \tag{30}$$

where $p\in\{0,\ldots,m+1\}$. In this expression sn is a Jacobi elliptic function (see for instance Chapter 22 in Olver *et al.* (2010) [25]).

Byrd & Friedman (1954) [5] give a procedure for computing (30) for all $m\in\mathbf{N}_0$. These integrals will be given in terms of complete elliptic integrals of the first three kinds (see the discussion in Byrd & Friedman (1954) [5], p. 201, 204, and p. 205). To this effect, we have the following definitions from (253.11) in Byrd & Friedman (1954) [5], namely

$$\alpha^2 = \frac{c-d}{b-d}<1,$$

$$\alpha_1^2 = \frac{b(c-d)}{c(b-d)},$$

$$g = \frac{2}{\sqrt{(a-c)(b-d)}},$$

$$\varphi = \sin^{-1}\sqrt{\frac{(b-d)(c-y)}{(c-d)(b-y)}},$$

$$u_1 = F(\varphi,k),$$

$$k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)},$$

with $k^2<\alpha^2$. For our specific choices in (29), these reduce to

$$\alpha^2 = \frac{2B}{A+B-1},$$

$$\alpha_1^2 = \frac{2(A-1)}{A+B-1},$$

$$g = \frac{2B}{\sqrt{(A+B-1)(A-B+1)}},$$

$$k^2 = \frac{4B}{(A+B-1)(A-B+1)},$$

$$\varphi = \frac{\pi}{2},$$

and

$$u_1 = K(k).$$

Specific cases include

$$\int_y^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = gK(k)$$

(Byrd & Friedman (1954) [5], (340.00)) and

$$\int_y^c \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{cg}{\alpha^2}\left[\alpha_1^2 K(k) + (\alpha^2-\alpha_1^2)\Pi(\alpha,k)\right]$$

(Byrd & Friedman (1954) [5], (340.01)).

In general we have

$$\int_y^c \frac{x^p\,dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{c^p g \alpha_1^{2p} p!}{\alpha^{2p}} \sum_{j=0}^{p} \frac{(\alpha^2-\alpha_1^2)^j}{\alpha_1^{2j} j!(p-j)!} V_j$$

(Byrd & Friedman (1954) [5], (340.04)), where

$$V_0 = K(k),$$

$$V_1 = \Pi(\alpha,k),$$

$$V_2 = \frac{1}{2(\alpha^2-1)(k^2-\alpha^2)}\left[(k^2-\alpha^2)K(k) + \alpha^2 E(k) + (2\alpha^2k^2 + 2\alpha^2 - \alpha^4 - 3k^2)\Pi(\alpha,k)\right],$$

and larger values of V_j can be computed using the following recurrence relation

$$\begin{aligned} V_{m+3} = {} & \frac{1}{2(m+2)(1-\alpha^2)(k^2-\alpha^2)} \\ & \times[(2m+1)k^2V_m + 2(m+1)(\alpha^2k^2+\alpha^2-3k^2)V_{m+1} \\ & +(2m+3)(\alpha^4-2\alpha^2k^2-2\alpha^2+3k^2)V_{m+2}] \end{aligned}$$

(see Byrd & Friedman (1954) [5], (336.00–03)). For instance,

$$\begin{aligned} & \int_y^c \frac{x^2\,dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} \\ & = \frac{c^2 g}{\alpha^4}\left[\alpha_1^4 K(k) + 2\alpha_1^2(\alpha^2-\alpha_1^2)\Pi(\alpha,k) + (\alpha^2-\alpha_1^2)^2 V_2\right] \end{aligned}$$

(see Byrd & Friedman (1954) [5], (340.02)).

In general, the Fourier coefficients for $\mathfrak{h}^3$ will be given in terms of complete elliptic integrals of the first three kinds. Let's directly compute the $m = 0$ component, in which (28) reduces to

$$\mathsf{H}_0^{1/2}(r, r', \theta, \theta') = -1 + \frac{1}{\pi}\int_{-1}^{1} \frac{(A/B - x)\,dx}{\sqrt{(1-x)(1+x)\left(x - \frac{A+1}{B}\right)\left(x - \frac{A-1}{B}\right)}}.$$

Therefore using the above formulae, we have

$$\begin{aligned}
\mathfrak{h}_3|_{m=0} &= \mathsf{H}_0^{1/2}(r, r', \theta, \theta') \\
&= -1 + \frac{2K(k)}{\pi\sqrt{(A-B+1)(A+B-1)}} + \frac{2(A-B-1)\Pi(\alpha, k)}{\pi\sqrt{(A-B+1)(A+B-1)}} \\
&= -1 + \frac{2}{\pi}\left\{K(k) + \left[\cosh r \cosh r' - \sinh r \sinh r' \cos(\theta - \theta') - 1\right]\Pi(\alpha, k)\right\} \\
&\qquad \times \left[\cosh r \cosh r' - \sinh r \sinh r' \cos(\theta - \theta') + 1\right]^{-1/2} \\
&\qquad \times \left[\cosh r \cosh r' - \sinh r \sinh r' \cos(\theta + \theta') - 1\right]^{-1/2}.
\end{aligned}$$

Note that the Fourier coefficients

$$\mathfrak{h}^3|_{m=0} \to \mathfrak{g}^3|_{m=0},$$

in the limit as $r, r' \to 0^+$, where $\mathfrak{g}^3|_{m=0}$ is given in (24). This is expected since $\mathbf{H}_R^3$ is a manifold.

3.3. Fourier expansion for a fundamental solution of the Laplacian on $\mathbf{H}_R^d$

For the d-dimensional Riemannian manifold $\mathbf{H}_R^d$, with $d \geq 2$, one can expand a fundamental solution of the Laplace-Beltrami operator in an azimuthal Fourier series. One may Fourier expand, in terms of the azimuthal coordinate, a fundamental solution of the Laplace-Beltrami operator in any rotationally-invariant coordinate systems which admits solutions via separation of variables. In Euclidean space, there exist non-subgroup-type rotationally invariant coordinate systems which are separable for Laplace's equation. All separable coordinate systems for Laplace's equation in d-dimensional Euclidean space $\mathbf{R}^d$ are known. In fact, this is also true for separable coordinate systems on $\mathbf{H}_R^d$ (see Kalnins (1986) [19]). There has been considerable work done in two and three dimensions, however there still remains a lot of work to be done for a detailed analysis of fundamental solutions.

We define an unnormalized fundamental solution of Laplace's equation on the unit hyperboloid $\mathfrak{h}^d : (\mathbf{H}^d \times \mathbf{H}^d) \setminus \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbf{H}^d\} \to \mathbf{R}$ such that

$$\mathfrak{h}^d(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}') := \mathcal{I}_d(\rho(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}')) = \frac{2\pi^{d/2} R^{d-2}}{\Gamma(d/2)} \mathcal{H}_R^d(\mathbf{x}, \mathbf{x}'). \tag{31}$$

In our current azimuthal Fourier analysis, we therefore will focus on the relatively easier case of separable subgroup-type coordinate systems on $\mathbf{H}_R^d$, and specifically for geodesic polar coordinates. In these coordinates the Riemannian metric is given by (10) and we further restrict our attention by adopting standard geodesic polar coordinates (1).

In these coordinates would would like to expand a fundamental solution of Laplace's equation on the hyperboloid in an azimuthal Fourier series, namely

$$\mathfrak{h}^d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}')=\sum_{m=0}^{\infty}\cos(m(\phi-\phi'))\mathsf{H}_m^{d/2-1}(r,r',\theta_1,\ldots,\theta_{d-2},\theta_1',\ldots,\theta_{d-2}') \tag{32}$$

where $\mathsf{H}_m^{d/2-1}:[0,\infty)^2\times[0,\pi]^{2d-4}\to\mathbf{R}$ is defined such that

$$\mathsf{H}_m^{d/2-1}(r,r',\theta_1,\ldots,\theta_{d-2},\theta_1',\ldots,\theta_{d-2}'):=\frac{\epsilon_m}{\pi}\int_0^\pi \mathfrak{h}^d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}')\cos(m(\phi-\phi'))d(\phi-\phi') \tag{33}$$

(see for instance Cohl & Tohline (1999) [10]). According to Theorem 2.1 and (31), we may write $\mathfrak{h}^d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}')$ in terms of associated Legendre functions of the second kind as follows

$$\mathfrak{h}^d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}')=\frac{e^{-i\pi(d/2-1)}}{2^{d/2-1}\Gamma(d/2)\left(\sinh d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}')\right)^{d/2-1}}Q_{d/2-1}^{d/2-1}\left(\cosh d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}')\right). \tag{34}$$

By (3) we know that in any geodesic polar coordinate system

$$\cosh d(\widehat{\mathbf{x}},\widehat{\mathbf{x}}')=\cosh r\cosh r'-\sinh r\sinh r'\cos\gamma, \tag{35}$$

and therefore through (33), (34), and (35), in standard geodesic polar coordinates, the azimuthal Fourier coefficient can be given by

$$\begin{aligned}&\mathsf{H}_m^{d/2-1}(r,r',\theta_1,\ldots,\theta_{d-2},\theta_1',\ldots,\theta_{d-2}')\\ &\qquad=\frac{\epsilon_m e^{-i\pi(d/2-1)}}{2^{d/2-1}\pi\Gamma(d/2)}\int_0^\pi\frac{Q_{d/2-1}^{d/2-1}\left(A-B\cos\psi\right)\cos(m\psi)}{\left[(A-B\cos\psi)^2-1\right]^{(d-2)/4}}d\psi,\end{aligned} \tag{36}$$

where $\psi:=\phi-\phi'$, $A,B:[0,\infty)^2\times[0,\pi]^{2d-4}\to\mathbf{R}$ are defined through (9) and (35) as

$$A(r,r',\theta_1,\ldots,\theta_{d-2},\theta_1',\ldots,\theta_{d-2}'):=\cosh r\cosh r'\sum_{i=1}^{d-2}\cos\theta_i\cos\theta_i{}'\prod_{j=1}^{i-1}\sin\theta_j\sin\theta_j{}',$$

and

$$B(r,r',\theta_1,\ldots,\theta_{d-2},\theta_1',\ldots,\theta_{d-2}'):=\sinh r\sinh r'\prod_{i=1}^{d-2}\sin\theta_i\sin\theta_i{}'.$$

Even though (36) is a compact expression for the Fourier coefficient of a fundamental solution of Laplace's equation on $\mathbf{H}_R^d$ for $d\in\{2,3,4,\ldots\}$, it may be informative to use any of the representations of a fundamental solution of the Laplacian on $\mathbf{H}_R^d$ from Theorem 2.1 to express the Fourier coefficients. For instance if one uses the finite-summation expression in the odd-dimensions, on can write the Fourier coefficients as a linear combination of integrals of the form

$$\int_{-1}^{1}\frac{\left[(a+b)/2-x\right]^{2k-1}x^p dx}{(a-x)^{k-1}(b-x)^{k-1}\sqrt{(a-x)(b-x)(c-x)(x-d)}},$$

where $x=\cos\psi$, $k\in\{1,\ldots,(d-1)/2\}$, $p\in\{0,\ldots,m\}$, and we have used the nomenclature of section 3.2. This integral is a rational function of x multiplied by an inverse square-root of a quartic in x. Because of this and due to the limits of

integration, we see that by definition, these are all given in terms of complete elliptic integrals. The special functions which represent the azimuthal Fourier coefficients on $\mathbf{H}_R^d$ are unlike the odd-half-integer degree, integer-order, associated Legendre functions of the second kind which appear in Euclidean space $\mathbf{R}^d$ for d odd (see Cohl (2010) [6]; Cohl & Dominici (2010) [7]), in that they include complete elliptic integrals of the third kind (in addition to complete elliptic integrals of the first and second kind) (cf. section 3.2) in their basis functions. For $d \geq 2$, through (4.1) in Cohl & Dominici (2010) [7] and that $\mathbf{H}_R^d$ is a manifold (and therefore must locally represent Euclidean space), the functions $\mathsf{H}_m^{d/2-1}$ are generalizations of associated Legendre functions of the second kind with odd-half-integer degree and order given by either an odd-half-integer or an integer.

4. Gegenbauer expansion in geodesic polar coordinates

In this section we derive an eigenfunction expansion for a fundamental solution of Laplace's equation on the hyperboloid in geodesic polar coordinates for $d \in \{3, 4, \ldots\}$. Since the spherical harmonics for $d = 2$ are just trigonometric functions with argument given in terms of the azimuthal angle, this case has already been covered in section 3.1.

In geodesic polar coordinates, Laplace's equation is given by (cf. (11))

$$\Delta f = \frac{1}{R^2}\left[\frac{\partial^2 f}{\partial r^2} + (d-1)\coth r \frac{\partial f}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{\mathbf{S}^{d-1}}\right] f = 0, \qquad (37)$$

where $f : \mathbf{H}_R^d \to \mathbf{R}$ and $\Delta_{\mathbf{S}^{d-1}}$ is the corresponding Laplace-Beltrami operator on the $(d-1)$-dimensional unit sphere $\mathbf{S}^{d-1}$. Eigenfunctions $Y_l^K : \mathbf{S}^{d-1} \to \mathbf{C}$ of the Laplace-Beltrami operator $\Delta_{\mathbf{S}^{d-1}}$, where $l \in \mathbf{N}_0$ and K is a set of quantum numbers which label representations for l in separable subgroup type coordinate systems on $\mathbf{S}^{d-1}$ (i.e. angular momentum type quantum numbers, see Izmest'ev *et al.* (2001) [18]), are given by solutions to the eigenvalue problem (12).

In standard geodesic polar coordinates (1), $K = (k_1, \ldots, k_{d-3}, |k_{d-2}|) \in \mathbf{N}_0^{d-2}$ with $k_0 = l \geq k_1 \geq \ldots \geq k_{d-3} \geq |k_{d-2}| \geq 0$, and in particular $k_{d-2} \in \{-k_{d-3}, \ldots, k_{d-3}\}$. A positive fundamental solution $\mathcal{H}_R^d : (\mathbf{H}_R^d \times \mathbf{H}_R^d) \setminus \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbf{H}_R^d\} \to \mathbf{R}$ on the R-radius hyperboloid satisfies (13). The completeness relation for hyperspherical harmonics in standard hyperspherical coordinates is given by

$$\sum_{l=0}^{\infty}\sum_K Y_l^K(\theta_1, \ldots, \theta_{d-1})\overline{Y_l^K(\theta_1', \ldots, \theta_{d-1}')} = \frac{\delta(\theta_1 - \theta_1')\ldots\delta(\theta_{d-1} - \theta_{d-1}')}{\sin^{d-2}\theta_{d-1}' \ldots \sin\theta_2'},$$

where $K = (k_1, \ldots, k_{d-2})$ and $l = k_0 \in \mathbf{N}_0$. Therefore through (15), we can write

$$\delta_g(\mathbf{x}, \mathbf{x}') = \frac{\delta(r - r')}{R^d \sinh^{d-1} r'} \sum_{l=0}^{\infty}\sum_K Y_l^K(\theta_1, \ldots, \theta_{d-1})\overline{Y_l^K(\theta_1', \ldots, \theta_{d-1}')}. \qquad (38)$$

For fixed $r, r' \in [0, \infty)$ and $\theta_1', \ldots, \theta_{d-1}' \in [0, \pi]$, since $\mathcal{H}_R^d$ is harmonic on its domain, its restriction is in $C^2(\mathbf{S}^{d-1})$, and therefore has a unique expansion in hyperspherical

harmonics, namely

$$\mathcal{H}_R^d(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_K u_l^K(r,r',\theta_1',\ldots,\theta_{d-1}')Y_l^K(\theta_1,\ldots,\theta_{d-1}), \tag{39}$$

where $u_l^K : [0,\infty)^2 \times [0,\pi]^{d-1} \to \mathbf{C}$. If we substitute (38) and (39) into (13) and use (37) and (12), we obtain

$$\sum_{l=0}^{\infty}\sum_K Y_l^K(\theta_1,\ldots,\theta_{d-1})\left[\frac{d^2}{dr^2}+(d-1)\coth r\frac{d}{dr}-\frac{l(l+d-2)}{\sinh^2 r}\right]u_l^K(r,r',\theta_1',\ldots,\theta_{d-1}')$$
$$= \sum_{l=0}^{\infty}\sum_K Y_l^K(\theta_1,\ldots,\theta_{d-1})\overline{Y_l^K(\theta_1',\ldots,\theta_{d-1}')}\cdot\frac{\delta(r-r')}{R^{d-2}\sinh^{d-1}r'}. \tag{40}$$

This indicates that for $u_l : [0,\infty)^2 \to \mathbf{R}$,

$$u_l^K(r,r',\theta_1',\ldots,\theta_{d-1}') = u_l(r,r')\overline{Y_l^K(\theta_1',\ldots,\theta_{d-1}')}, \tag{41}$$

and from (39) the expression for a fundamental of the Laplace-Beltrami operator in hyperspherical coordinates on the hyperboloid is given by

$$\mathcal{H}_R^d(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}u_l(r,r')\sum_K Y_l^K(\theta_1,\ldots,\theta_{d-1})\overline{Y_l^K(\theta_1',\ldots,\theta_{d-1}')}. \tag{42}$$

The above expression can be simplified using the addition theorem for hyperspherical harmonics (see for instance Wen & Avery (1985) [30], section 10.2.1 in Fano & Rau (1996), Chapter 9 in Andrews, Askey & Roy (1999) [2] and especially Chapter XI in Erdélyi *et al.* Vol. II (1981) [12]), which is given by

$$\sum_K Y_l^K(\widehat{\mathbf{x}})\overline{Y_l^K(\widehat{\mathbf{x}}')} = \frac{\Gamma(d/2)}{2\pi^{d/2}(d-2)}(2l+d-2)C_l^{d/2-1}(\cos\gamma), \tag{43}$$

where γ is the angle between two arbitrary vectors $\widehat{\mathbf{x}},\widehat{\mathbf{x}}'\in\mathbf{S}^{d-1}$ given in terms of (2). The Gegenbauer polynomials $C_l^\mu : [-1,1]\to\mathbf{R}$, $l\in\mathbf{N}_0$, $\mathrm{Re}\,\mu > -1/2$, can be defined in terms of the Gauss hypergeometric function as

$$C_l^\mu(x) := \frac{(2\mu)_l}{l!}\,{}_2F_1\left(-l,l+2\mu;\mu+\frac{1}{2};\frac{1-x}{2}\right).$$

The above expression (42) can be simplified using (43), therefore

$$\mathcal{H}_R^d(\mathbf{x},\mathbf{x}') = \frac{\Gamma(d/2)}{2\pi^{d/2}(d-2)}\sum_{l=0}^{\infty}u_l(r,r')(2l+d-2)C_l^{d/2-1}(\cos\gamma). \tag{44}$$

Now we compute the exact expression for $u_l(r,r')$. By separating the angular dependence in (40) and using (41), we obtain the differential equation

$$\frac{d^2u_l}{dr^2}+(d-1)\coth r\frac{du_l}{dr}-\frac{l(l+d-2)u_l}{\sinh^2 r} = -\frac{\delta(r-r')}{R^{d-2}\sinh^{d-1}r'}. \tag{45}$$

Away from $r=r'$, solutions to the differential equation (45) must be given by solutions to the homogeneous equation, which are are given in section 2.2. Therefore, the solution to (45) is given by

$$u_l(r,r') = \frac{A}{(\sinh r\sinh r')^{d/2-1}}P_{d/2-1}^{-(d/2-1+l)}(\cosh r_<)Q_{d/2-1}^{d/2-1+l}(\cosh r_>), \tag{46}$$

such that $u_l(r,r')$ is continuous at $r=r'$, where $A\in\mathbf{R}$.

In order to determine the constant A, we first make the substitution

$$v_l(r,r') = (\sinh r \sinh r')^{(d-1)/2} u_l(r,r'). \tag{47}$$

This converts (45) into the following differential equation

$$\frac{\partial^2 v_l(r,r')}{\partial r^2} - \frac{1}{4}\left[\frac{(d-1+2l)(d-3+2l)}{\sinh^2 r} + (d-1)^2\right] v_l(r,r') = -\frac{\delta(r-r')}{R^{d-2}},$$

which we then integrate over r from $r'-\epsilon$ to $r'+\epsilon$, and take the limit as $\epsilon\to 0^+$. We are left with a discontinuity condition for the derivative of $v_l(r,r')$ with respect to r evaluated at $r=r'$, namely

$$\lim_{r\to 0^+} \frac{dv_l(r,r')}{dr}\bigg|_{r'-\epsilon}^{r'+\epsilon} = \frac{-1}{R^{d-2}}. \tag{48}$$

After inserting (46) with (47) into (48), substituting $z=\cosh r'$, evaluating at $r=r'$, and making use of the Wronskian relation (e.g. p. 165 in Magnus, Oberhettinger & Soni (1966) [21])

$$W\left\{P_\nu^{-\mu}(z), Q_\nu^\mu(z)\right\} = -\frac{e^{i\pi\mu}}{z^2-1},$$

which is equivalent to

$$W\left\{P_\nu^{-\mu}(\cosh r'), Q_\nu^\mu(\cosh r')\right\} = -\frac{e^{i\pi\mu}}{\sinh^2 r'},$$

we obtain

$$A = \frac{e^{-i\pi(d/2-1+l)}}{R^{d-2}},$$

and hence

$$u_l(r,r') = \frac{e^{-i\pi(d/2-1+l)}}{R^{d-2}(\sinh r\sinh r')^{d/2-1}} P_{d/2-1}^{-(d/2-1+l)}(\cosh r_<) Q_{d/2-1}^{d/2-1+l}(\cosh r_>),$$

and therefore through (44), we have

$$\mathcal{H}_R^d(\mathbf{x},\mathbf{x}') = \frac{\Gamma(d/2)}{2\pi^{d/2}R^{d-2}(d-2)} \frac{e^{-i\pi(d/2-1)}}{(\sinh r\sinh r')^{d/2-1}}$$
$$\times \sum_{l=0}^\infty (-1)^l (2l+d-2) P_{d/2-1}^{-(d/2-1+l)}(\cosh r_<) Q_{d/2-1}^{d/2-1+l}(\cosh r_>) C_l^{d/2-1}(\cos\gamma). \tag{49}$$

As an alternative check of our derivation, we can do the asymptotics for the product of associated Legendre functions $P_{d/2-1}^{-(d/2-1+l)}(\cosh r_<) Q_{d/2-1}^{d/2-1+l}(\cosh r_>)$ in (49) as $r,r'\to 0^+$. The appropriate asymptotic expressions for P and Q respectively can be found on p. 171 and p. 173 in Olver (1997) [24]. For the associated Legendre function of the first kind there is

$$P_\nu^{-\mu}(z) \sim \frac{[(z-1)/2]^{\mu/2}}{\Gamma(\mu+1)},$$

as $z \to 1$, $\mu \neq -1, -2, \ldots$, and for the associated Legendre function of the second kind there is

$$Q_\nu^\mu(z) \sim \frac{e^{i\pi\mu}\Gamma(\mu)}{2\left[(z-1)/2\right]^{\mu/2}},$$

as $z \to 1^+$, Re $\mu > 0$, and $\nu+\mu \neq -1, -2, -3, \ldots$. To second order the hyperbolic cosine is given by $\cosh r \simeq 1+r^2/2$. Therefore to lowest order we can insert $\cosh r_< \simeq 1+r_<^2/2$ and $\cosh r_> \simeq 1 + r_>^2/2$ into the above expressions yielding

$$P_{d/2-1}^{-(d/2-1+l)}(\cosh r_<) \sim \frac{(r_</2)^{d/2-1+l}}{\Gamma(d/2+l)},$$

and

$$Q_{d/2-1}^{d/2-1+l}(\cosh r_>) \sim \frac{e^{i\pi(d/2-1+l)}\Gamma(d/2-1+l)}{2(r_>/2)^{d/2-1+l}},$$

as $r, r' \to 0^+$. Therefore the asymptotics for the product of associated Legendre functions in (49) is given by

$$P_{d/2-1}^{-(d/2-1+l)}(\cosh r_<) Q_{d/2-1}^{d/2-1+l}(\cosh r_>) \sim \frac{e^{i\pi(d/2-1+l)}}{2l+d-2}\left(\frac{r_<}{r_>}\right)^{l+d/2-1} \tag{50}$$

(the factor $2l+d-2$ is a term which one encounters regularly with hyperspherical harmonics). Gegenbauer polynomials obey the following generating function

$$\frac{1}{(1+z^2-2zx)^\mu} = \sum_{l=0}^{\infty} C_l^\mu(x) z^l, \tag{51}$$

where $x \in [-1, 1]$ and $|z| < 1$ (see for instance, p. 222 in Magnus, Oberhettinger & Soni (1966) [21]). The generating function for Gegenbauer polynomials (51) can be used to expand a fundamental solution of Laplace's equation in Euclidean space $\mathbf{R}^d$ (for $d \geq 3$, cf. Theorem 3.1) in hyperspherical coordinates, namely

$$\frac{1}{\|\mathbf{x}-\mathbf{x}'\|^{d-2}} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+d-2}} C_l^{d/2-1}(\cos\gamma), \tag{52}$$

where γ was defined in (43). Using (52) and Theorem 3.1, since $\mathcal{H}_R^d \to \mathcal{G}^d$, $\sinh r, \sinh r' \to r, r'$ and (50) is satisfied to lowest order as $r, r' \to 0^+$, we see that (49) obeys the correct asymptotics and our fundamental solution expansion locally reduces to the appropriate expansion for Euclidean space, as it should since $\mathbf{H}_R^d$ is a manifold.

Note that (49) can be further expanded over the remaining $(d-2)$-quantum numbers in K in terms of a simply separable product of normalized harmonics $Y_l^K(\widehat{\mathbf{x}})\overline{Y_l^K(\widehat{\mathbf{x}}')}$, where $\widehat{\mathbf{x}}, \widehat{\mathbf{x}}' \in \mathbf{S}^{d-1}$, using the addition theorem for hyperspherical harmonics (43) (see Cohl (2010) [6] for several examples).

It is intriguing to observe how one might obtain the Fourier expansion for $d = 2$ (23) from the the expansion (49), which is strictly valid for $d \geq 3$. If one makes the substitution $\mu = d/2 - 1$ in (49) then we obtain the following conjecture (which

matches up to the generating function for Gegenbauer polynomials in the Euclidean limit $r, r' \to 0^+$)

$$\frac{1}{\sinh^\mu \rho} Q_\mu^\mu(\cosh \rho) = \frac{2^\mu \Gamma(\mu+1)}{(\sinh r \sinh r')^\mu} \times \sum_{n=0}^{\infty} (-1)^n \frac{n+\mu}{\mu} P_\mu^{-(\mu+n)}(\cosh r_<) Q_\mu^{\mu+n}(\cosh r_>) C_n^\mu(\cos\gamma), \quad (53)$$

for all $\mu \in \mathbf{C}$ such that $\mathrm{Re}\,\mu > -1/2$. If we take the limit as $\mu \to 0$ in (53) and use

$$\lim_{\mu\to 0} \frac{n+\mu}{\mu} C_n^\mu(x) = \epsilon_n T_n(x) \quad (54)$$

(see for instance (6.4.13) in Andrews, Askey & Roy (1999) [2]), where $T_n : [-1,1] \to \mathbf{R}$ is the Chebyshev polynomial of the first kind defined as $T_l(x) := \cos(l \cos^{-1} x)$, then we obtain the following formula

$$\frac{1}{2} \log \frac{\cosh\rho + 1}{\cosh\rho - 1} = \sum_{n=0}^{\infty} \epsilon_n (-1)^n P_0^{-n}(\cosh r_<) Q_0^n(\cosh r_>) \cos(n(\phi - \phi')),$$

where $\cosh\rho = \cosh r \cosh r' - \sinh r \sinh r' \cos(\phi - \phi')$. By taking advantage of the following formulae

$$P_0^{-n}(z) = \frac{1}{n!} \left[\frac{z-1}{z+1}\right]^{n/2}, \quad (55)$$

for $n \ge 0$,

$$Q_0(z) = \frac{1}{2} \log \frac{z+1}{z-1} \quad (56)$$

(see (8.4.2) in Abramowitz & Stegun (1972) [1]), and

$$Q_0^n(z) = \frac{1}{2} (-1)^n (n-1)! \left\{ \left[\frac{z+1}{z-1}\right]^{n/2} - \left[\frac{z-1}{z+1}\right]^{n/2} \right\}, \quad (57)$$

for $n \ge 1$ then (23) is reproduced. The representation (55) follows easily from the Gauss hypergeometric representation of the associated Legendre function of the first kind (see (8.1.2) in Abramowitz & Stegun (1972) [1])

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left[\frac{z+1}{z-1}\right]^{\mu/2} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right). \quad (58)$$

One way to derive the representation of the associated Legendre function of second kind (57) is to use Whipple formula for associated Legendre functions (cf. (8.2.7) in Abramowitz & Stegun (1972) [1])

$$Q_\nu^\mu(z) = \sqrt{\pi} 2\Gamma(\nu+\mu+1)(z^2-1)^{-1/4} e^{i\pi\mu} P_{-\mu-1/2}^{-\nu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right),$$

and (8.6.9) in Abramowitz & Stegun (1972) [1], namely

$$P_\nu^{-1/2}(z) = \sqrt{\frac{2}{\pi}} \frac{(z^2-1)^{-1/4}}{2\nu+1} \left\{ \left[z + \sqrt{z^2-1}\right]^{\nu+1/2} - \left[z + \sqrt{z^2-1}\right]^{-\nu-1/2} \right\},$$

for $\nu \ne -1/2$.

4.1. Addition theorem for the azimuthal Fourier coefficient on $\mathbf{H}_R^3$

One can compute addition theorems for the azimuthal Fourier coefficients of a fundamental solution for Laplace's equation on $\mathbf{H}_R^d$ for $d \geq 3$ by relating directly obtained Fourier coefficients to the expansion over hyperspherical harmonics for the same fundamental solution. By using the expansion of $\mathcal{H}_R^d(\mathbf{x}, \mathbf{x}')$ in terms of Gegenbauer polynomials (49) in combination with the addition theorem for hyperspherical harmonics (43) expressed in, for instance, one of Vilenkin's polyspherical coordinates (see section IX.5.2 in Vilenkin (1968) [29]; Izmest'ev *et al.* (1999,2001) [17, 18]), one can obtain through series rearrangement a multi-summation expression for the azimuthal Fourier coefficients. Vilenkin's polyspherical coordinates are simply subgroup-type coordinate systems which parametrize points on $\mathbf{S}^{d-1}$ (for a detailed discussion of these coordinate systems see chapter 4 in Cohl (2010) [6]). In this section we will give an explicit example of just such an addition theorem on $\mathbf{H}_R^3$.

The azimuthal Fourier coefficients on $\mathbf{H}_R^3$ expressed in standard hyperspherical coordinates (1) are given by the functions $\mathsf{H}_m : [0, \infty)^2 \times [0, \pi]^2 \to \mathbf{R}$ which is defined by (27). By expressing (49) in $d = 3$ we obtain

$$\mathcal{H}_R^3(\mathbf{x}, \mathbf{x}') = \frac{-i}{4\pi R\sqrt{\sinh r \sinh r'}} \times \sum_{l=0}^{\infty} (-1)^l (2l+1) P_{1/2}^{-(1/2+l)}(\cosh r_<) Q_{1/2}^{1/2+l}(\cosh r_>) P_l(\cos\gamma), \tag{59}$$

where $P_l : [-1, 1] \to \mathbf{R}$ is the Legendre polynomial defined by $P_l(x) = C_l^{1/2}(x)$, or through (58) with $\mu = 0$ and $\nu \in \mathbf{N}_0$. By using the addition theorem for hyperspherical harmonics (43) with $d = 3$ using $(\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$ to parametrize points on $\mathbf{S}^2$, we have since the normalized spherical harmonics are

$$Y_{l,m}(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi},$$

the addition theorem for spherical harmonics, namely

$$P_l(\cos\gamma) = \sum_{m=-l}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta) P_l^m(\cos\theta') e^{im(\phi-\phi')}, \tag{60}$$

where $\cos\gamma$ is given by (25). By combining (59) and (60), reversing the order of the two summation symbols, and comparing the result with (26) we obtain the following single summation addition theorem for the azimuthal Fourier coefficients of a fundamental solution of Laplace's equation on $\mathbf{H}_R^3$, namely since $\mathfrak{h}^3 = 4\pi R \mathcal{H}_R^3$,

$$\mathsf{H}_m^{1/2}(r, r', \theta, \theta') = \frac{-i\epsilon_m}{\sqrt{\sinh r \sinh r'}} \sum_{l=|m|}^{\infty} (-1)^l (2l+1) \frac{(l-m)!}{(l+m)!} \times P_l^m(\cos\theta) P_l^m(\cos\theta') P_{1/2}^{-(1/2+l)}(\cosh r_<) Q_{1/2}^{1/2+l}(\cosh r_>).$$

This addition theorem reduces to the corresponding result ((2.4) in Cohl *et al.* (2001) [9]) in the Euclidean $\mathbf{R}^3$ limit as $r, r' \to 0^+$.

5. Discussion

Re-arrangement of the multi-summation expressions in section 4 is possible through modification of the order in which the countably infinite space of quantum numbers is summed over in a standard hyperspherical coordinate system, namely

$$\sum_{l=0}^{\infty}\sum_{K} = \sum_{l=0}^{\infty}\sum_{k_1=0}^{l}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{d-4}=0}^{k_{d-5}}\sum_{k_{d-3}=0}^{k_{d-4}}\sum_{k_{d-2}=-k_{d-3}}^{k_{d-3}}$$
$$= \sum_{k_{d-2}=-\infty}^{\infty}\sum_{k_{d-3}=|k_{d-2}|}^{\infty}\sum_{k_{d-4}=k_{d-2}}^{\infty}\cdots\sum_{k_2=k_3}^{\infty}\sum_{k_1=k_2}^{\infty}\sum_{k_0=k_1}^{\infty}.$$

Similar multi-summation re-arrangements have been accomplished previously for azimuthal Fourier coefficients of fundamental solutions for the Laplacian in Euclidean space (see for instance Cohl *et al.* (2000) [11]; Cohl *et al.* (2001) [9]). Comparison of the azimuthal Fourier expansions in section 3 (and in particular (36)) with re-arranged Gegenbauer expansions in section 4 (and in particular (49)) will yield new addition theorems for the special functions representing the azimuthal Fourier coefficients of a fundamental solution of the Laplacian on the hyperboloid. These implied addition theorems will provide new special function identities for the azimuthal Fourier coefficients, which are hyperbolic generalizations of particular associated Legendre functions of the second kind. In odd-dimensions, these special functions reduce to toroidal harmonics.

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